MEASURE AND INTEGRATION 2018-2019 – FINAL EXAM

Instructor: Daniel Valesin

1. **[15 points]** Give an example of a Dynkin system that is not a σ-algebra.

2. **[15 points]** Show that if $f : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous (that is, there exists $c > 0$ such that $|f(x) - f(y)| \leq c|x - y|$ for any x, y) and $A \subset \mathbb{R}$ has $m^*(A) = 0$, then $m^*(f(A)) = 0$ (here m^* denotes the Lebesgue outer measure on $\mathbb{R}$).

3. **[15 points]** Evaluate
$$\lim_{n\to\infty} \int_0^\infty \frac{\sin(nx)}{\sqrt{x}} e^{-nx}\, dx.$$

4. **[15 points]** Prove that $\mathcal{B}^m \otimes \mathcal{B}^n = \mathcal{B}^{m+n}$ (here $\mathcal{B}^d$ denotes the Borel σ-algebra in $\mathbb{R}^d$).

5. **[15 points]** Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and $f \in \mathcal{L}^p(\Omega)$, with $p \in (1, \infty)$. Let q be the conjugate exponent of p and define
$$F : \mathcal{L}^q(\Omega) \to \mathbb{R}, \quad F(g) = \int_\Omega fg\, d\mu.$$
Show that F is a bounded linear functional. Show that $\|F\| = \|f\|_p$.

6. (a) **[8 points]** Prove or give a counterexample to the following statement. Let $(\Omega, \mathcal{F})$ be a measurable space and μ, ν be σ-finite measures on $\mathcal{F}$ with $\mu \ll \nu$. Then, for any $\varepsilon > 0$ there exists $\delta > 0$ such that, if $A \in \mathcal{F}$ has $\nu(A) < \delta$, then $\mu(A) < \varepsilon$.

   (b) **[7 points]** Let $f : \mathbb{R} \to \mathbb{R}$ be non-negative and integrable (with respect to Lebesgue measure). Let
$$F(x) = \int_{-\infty}^x f(t)\, dt, \quad -\infty < x < \infty.$$
Show that for any $\varepsilon > 0$ there exists $\delta > 0$ such that, for any $n \in \mathbb{N}$ and any real numbers
$$a_1 < b_1 < a_2 < b_2 < \cdots < a_n < b_n$$
with $\sum_{i=1}^n (b_i - a_i) < \delta$, we have $\sum_{i=1}^n |F(b_i) - F(a_i)| < \varepsilon$.